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# Solvability of convolution equations in Gevrey classes of Roumieu type

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Let  $E^{\{d\}}(\mathbb{R}^N)$  denote the Gevrey class of exponent  $d > 1$  of Roumieu type. Cattabriga [2] has shown that in general not every constant coefficient linear partial differential equation  $P(D)f = g$  has a solution  $f \in E^{\{d\}}(\mathbb{R}^N)$  for arbitrary  $g \in E^{\{d\}}(\mathbb{R}^N)$ . In [9], Zampieri gave sufficient conditions on  $P(D)$  for the solvability of the above equation in  $E^{\{d\}}(\mathbb{R}^N)$ . He used ideas of Hörmander [4], who has characterized those  $P(D)$  which act surjectively on the real-analytic functions on  $\mathbb{R}^N$ , i.e. on  $E^{\{1\}}(\mathbb{R}^N)$ .

Recently, we have been able to characterize those convolution operators  $T_\mu$  on the ultradifferentiable functions  $E_{\{\omega\}}(\mathbb{R})$  of Roumieu type which are surjective. In the present note we state this result for the Gevrey classes  $E^{\{d\}}(\mathbb{R})$  and give a rough sketch of its proof. For details we refer to our forthcoming paper [1].

1. Definition. For  $d > 1$  we define the Gevrey class  $E^{\{d\}}(\mathbb{R})$  of Roumieu type and exponent  $d$  by

$E^{\{d\}}(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}) \mid \text{for each compact set } K \subset \mathbb{R} \text{ there exists } h > 0:$

$$\sup_{x \in K} \sup_{p \in \mathbb{N}_0} \frac{|f^{(p)}(x)|}{h^p (p!)^d} < \infty\}.$$

We endow  $E^{\{d\}}(\mathbb{R})$  with its usual topology, which is obtained by taking first the inductive limit over  $h > 0$  and then the projective limit of all compact sets  $K$  in  $\mathbb{R}$  (see Komatsu [5], 2.5). Furthermore we define

$$\mathcal{D}^{\{d\}}(\mathbb{R}) := \varinjlim_{n \rightarrow} \mathcal{D}^{\{d\}}[-n, n],$$

where

$$\mathcal{D}^{\{d\}}[-n, n] := \{f \in E^{\{d\}}(\mathbb{R}) \mid \text{Supp}(f) \subset [-n, n]\},$$

endowed with the topology induced by  $E^{\{d\}}(\mathbb{R})$ .

## 2. Convolution operators

For  $d > 1$  and  $\mu \in E^{\{d\}}(\mathbb{R})$ , we define the convolution operator  $T_\mu$  on  $E^{\{d\}}(\mathbb{R})$ , which is induced by  $\mu$ , in the following way

$$T_\mu(f) : x \mapsto \langle \mu_y, f(x-y) \rangle, \quad f \in E^{\{d\}}(\mathbb{R}).$$

It is easy to check that  $T_\mu$  is a continuous linear endomorphism of  $E^{\{d\}}(\mathbb{R})$ .

For  $v \in \mathcal{D}^{\{d\}}(\mathbb{R})$ , we define  $\mu * v$  by

$$\langle \mu * v, \varphi \rangle := \langle v, \check{\mu} * \varphi \rangle, \quad \varphi \in \mathcal{D}^{\{d\}}(\mathbb{R}),$$

where  $\langle \check{\mu}, g \rangle := \langle \mu_x, g(-x) \rangle$  for  $g \in E^{\{d\}}(\mathbb{R})$ . It is easy to see that  $\mu * v$  is in  $\mathcal{D}^{\{d\}}(\mathbb{R})$ .

Furthermore we define the Fourier-Laplace transform  $\hat{\mu}$  of  $\mu$  by

$$\hat{\mu} : \mathbb{C} \rightarrow \mathbb{C}, \quad \hat{\mu}(z) := \langle \mu_x, e^{-ixz} \rangle,$$

and we put

$$V(\hat{\mu}) := \{z \in \mathbb{C} \mid \hat{\mu}(z) = 0\}.$$

Using the notation introduced above, we can state our main result:

**3. Theorem.** For  $d > 1$  and  $\mu \in E^{\{d\}}(\mathbb{R})$ , the following conditions are equivalent:

(1)  $T_\mu : E^{\{d\}}(\mathbb{R}) \rightarrow E^{\{d\}}(\mathbb{R})$  is surjective.

(2)  $\mu$  satisfies (i) and (ii):

(i) there exists  $v \in \mathcal{D}^{\{d\}}(\mathbb{R})$  with  $\mu * v = \delta$ , i.e.  $T_\mu$  admits a fundamental solution

(ii)  $V(\hat{\mu})$  can be decomposed as  $V(\hat{\mu}) = V_0 \dot{\cup} V_1$  with

$$\lim_{\substack{|z| \rightarrow \infty \\ z \in V_0}} \frac{|\operatorname{Im} z|}{|z|^{1/d}} = 0 \quad \text{and} \quad \liminf_{\substack{|z| \rightarrow \infty \\ z \in V_1}} \frac{|\operatorname{Im} z|}{|z|^{1/d}} > 0.$$

4. Remark. (a) In Theorem 3, condition (2)(ii) can be replaced by each of the following conditions (iii) or (iv)

(iii)  $\ker T_\mu$  is reflexive (or bornological, or quasi-barrelled)

(iv)  $\ker T_\mu$  is linear topologically isomorphic to  $X \times Y$  for some nuclear Fréchet space  $X$  and some (DFN)-space  $Y$ .

(b) In Theorem 3.(2)(ii) the term  $|z|^{1/d}$  can be replaced by  $|\operatorname{Re} z|^{1/d}$ .

To sketch the proof of Theorem 3, fix  $d > 1$  and  $n \in \mathbb{N}$ . Then put

$$E^{\{d\}}(n) := \{f \in E^{\{d\}}(\mathbb{R}) \mid f(x) = 0 \text{ for all } x \in [-n, n]\}$$

and define  $E_n := E^{\{d\}}(\mathbb{R}) / E^{\{d\}}(n)$ .

Next fix  $\mu \in E^{\{d\}}(\mathbb{R})'$  and choose  $m \in \mathbb{N}$  with  $\operatorname{Supp}(\mu) \subset [-m, m]$ . Then it is easy to check that  $T_\mu$  induces for each  $n \in \mathbb{N}$  a continuous linear map

$$\tau_n : E_{n+m} \rightarrow E_n \text{ by } \tau_n(f + E^{\{d\}}(n+m)) := T_\mu(f) + E^{\{d\}}(n).$$

Let  $K(\mu, d)$  denote the projective spectrum consisting of the spaces  $(\ker \tau_n)_{n \in \mathbb{N}}$  and the natural restriction maps. Using properties of the projective limit functor of Palamodov [7], one can see that the following holds:

5. Lemma. For  $d > 1$  let  $\mu \in E^{\{d\}}(\mathbb{R})'$  be given. Assume that there exists  $\nu \in \mathcal{D}^{\{d\}}(\mathbb{R})'$  with  $\mu * \nu = \delta$ . Then  $T_\mu$  is surjective on  $E^{\{d\}}(\mathbb{R})$  if and only if  $\operatorname{Proj}^1 K(\mu, d) = 0$ .

To evaluate the condition  $\operatorname{Proj}^1 K(\mu, d) = 0$  we use a recent result of Vogt [8], by which  $\operatorname{Proj}^1 \Lambda(A) = 0$  is characterized for certain countable projective spectra  $\Lambda(A)$  of (DF)-sequence spaces. To state his result, we introduce the following notation.

6. Definition. Assume that  $A = (a_{j,k,m})_{j,k,m \in \mathbb{N}}$  satisfies the following conditions for all  $j,k,m \in \mathbb{N}$ :

$$(1) a_{j,k,m} > 0 \quad (2) a_{j,k,m} \leq a_{j,k+1,m} \quad (3) a_{j,k,m} \geq a_{j,k,m+1}.$$

Then we fix  $k,m \in \mathbb{N}$  and define

$$\lambda(k,m) := \{x \in \mathbb{C}^{\mathbb{N}} \mid \|x\|_{k,m} := \sum_{j=1}^{\infty} |x_j| a_{j,k,m} < \infty\}.$$

Then  $(\lambda(k,m), \|\cdot\|_{k,m})$  is a Banach space and because of (3) the inclusion map  $j_{m,m+1}^k : \lambda(k,m) \rightarrow \lambda(k,m+1)$  is continuous. Hence we can define

$$\lambda(k) := \text{ind}_{m \rightarrow \infty} \lambda(k,m).$$

Then (2) implies that the inclusion  $i_{k+1}^k : \lambda(k+1) \rightarrow \lambda(k)$  is continuous. By  $\Lambda(A)$  we denote the projective spectrum  $(\lambda(k), i_{k+1}^k)_{k \in \mathbb{N}}$ .

7. Example. Let  $\alpha = (\alpha_j)_{j \in \mathbb{N}}$  and  $\beta = (\beta_j)_{j \in \mathbb{N}}$  be sequences of non-negative numbers with  $\lim_{j \rightarrow \infty} \beta_j = \infty$ . Then

$$A_{\alpha,\beta} := (\exp(k\alpha_j + \frac{1}{m}\beta_j))_{j,k,m \in \mathbb{N}}$$

obviously satisfies the conditions (1)-(3) in Definition 6. In this case,  $\lambda(k)$  is a (DFS)-space for each  $k \in \mathbb{N}$  and we shall denote  $\text{proj}_{\leftarrow k} \lambda(k)$  by  $\lambda(\alpha, \beta)$ .

The result of Vogt [8], mentioned above, can now be stated as follows:

8. Theorem (Vogt [8]). Let  $A = (a_{j,k,m})_{j,k,m \in \mathbb{N}}$  satisfy the conditions in 6. and let  $\Lambda(A)$  denote the projective spectrum defined in 6. Then the following conditions are equivalent:

$$(1) \text{Proj}^1 \Lambda(A) = 0$$

$$(2) \forall \mu \in \mathbb{N} \exists n, k \in \mathbb{N} \forall m, L \in \mathbb{N} \exists N, S \in \mathbb{N} \forall j \in \mathbb{N} :$$

$$\frac{1}{a_{j,k,m}} \leq S \max \left( \frac{1}{a_{j,L,N}}, \frac{1}{a_{j,\mu,n}} \right).$$

The application of Theorem 8 is possible because of the following result:

9. Theorem (Meise [6]). For  $d > 1$  let  $\mu \in E^{\{d\}}(\mathbb{R})'$  be given. Assume there exists  $v \in \mathcal{D}^{\{d\}}(\mathbb{R})'$  with  $\mu * v = \delta$  and  $\dim \ker T_\mu = \infty$ . Then  $\ker T_\mu$  is linear topologically isomorphic to  $\lambda(\alpha, \beta)$ , where  $\alpha = (|\operatorname{Im} a_j|)_{j \in \mathbb{N}}$ ,  $\beta = (|a_j|^{1/d})_{j \in \mathbb{N}}$  for a sequence  $(a_j)_{j \in \mathbb{N}}$  which counts the zeros of  $\hat{\mu}$  with multiplicities.

Sketch of proof of Theorem 3: Arguments of Ehrenpreis [3] can be used to show that the surjectivity of  $T_\mu$  implies the existence of a fundamental solution for  $T_\mu$ . For the rest of the proof we can assume w.l.o.g. that  $\dim \ker T_\mu = \infty$ . Then Theorem 9 can be used to show that the projective spectra  $\Lambda(A_{\alpha, \beta})$  and  $K(\mu, d)$  are equivalent. By the properties of the projective limit functor, this implies  $\operatorname{Proj}^1 K(\mu, d) = \operatorname{Proj}^1 \Lambda(A_{\alpha, \beta})$ . Hence the result follows from Lemma 5 by evaluating condition 8.(2) for  $A_{\alpha, \beta}$ .

10. Example. There exists an ultradifferential operator

$\mu \in \bigcap_{d>1} E^{\{d\}}(\mathbb{R})'$  with the following properties:

- (1)  $V(\hat{\mu}) = \{\pm e^{j \pm i e^k} \mid j, k \in \mathbb{N}\}$
- (2)  $T_\mu : E^{\{d\}}(\mathbb{R}) \rightarrow E^{\{d\}}(\mathbb{R})$  admits a fundamental solution for each  $d > 1$
- (3) for each  $d > 1$ ,  $T_\mu : E^{\{d\}}(\mathbb{R}) \rightarrow E^{\{d\}}(\mathbb{R})$  is not surjective.

However, for each  $d > 1$ ,  $T_\mu : E^{(d)}(\mathbb{R}) \rightarrow E^{(d)}(\mathbb{R})$  is surjective, where  $E^{(d)}(\mathbb{R})$  denotes the Gevrey class of Beurling type of exponent  $d$ .

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